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Short Communication

Free vibration of circular and annular plates with variable thickness and different combinations of boundary conditions

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Abstract

In this paper, the first nine frequency parameters of circular and annular plates with variable thickness and combined boundary conditions are computed for different thickness to radius ratios. Several combined boundary conditions are considered for inner and outer edges. These are free, soft and hard simply supported, and clamped boundary conditions. Three-dimensional elasticity theory is used. Results of this paper are compared with works of other authors and the results of finite element method analysis which are in agreement. Data for thick circular and annular plates with linear and parabolic thickness variation and different combinations are presented for the first time.

1. Introduction

From a practical viewpoint, knowledge of plate vibrations is very important to structural designers. The majority of previous works in the field are based on two-dimensional (2-D) theories. For thin plates, classical plate theory can predict natural frequencies, but as the plate thickness increases, the computed frequencies are over-predicted. To improve the results, Mindlin theory, another 2-D method of analysis, can be used [1]. But results of Mindlin theory are valid only for lower frequency flexural modes of vibration of moderately thick plates. Higher-order shear deformation plate theory can be used to account for the effects of the shear deformation [2].

In three-dimensional (3-D) methods, no priori assumption is made about the plate. This method provides full elasticity solutions of the vibration problems. The results are completely accurate and no mode is missed. Not only can the 3-D solutions provide accurate solutions to the problems, but are also useful for assessing solutions of the 2-D theories.

Free vibration analysis of thick circular and annular plates with variable thickness is performed, and the results and details are reported in this paper. For very thick plates, the descriptive terminology "circular and annular plates" loses meaning, and other terms such as "solid and hollow cylinders" make more sense. For a review of research on thick plate vibration see Ref. [3]. Circular and annular plates with constant thickness

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have previously been studied by several investigators, see e.g. Ref. [4]. Indeed, creating appropriate variations in the plate's thickness could result in better mechanical behaviors of the plates. For example, efficiency of the plate is improved in situations where there is buckling, bending, and vibration. Vibration analysis of the plates with variable thickness was also carried out by using 2-D methods [5]. As far as the authors could ascertain, the first solution of 3-D elasticity vibrations of circular plates is due to Hutchinson [6,7]. Recently, some authors have used 3-D theories for vibration analysis of circular and annular plates [8–14].

The objective of the current paper is to detail the results of the work undertaken to study the vibrations of thick circular and annular plate, with variable thickness and different combinations of inner and outer boundary conditions. The plates were assumed to have linear and nonlinear thickness variation. Free, soft simply supported, hard simply supported, and clamped boundary conditions were considered for inner and outer edges. For several different combinations of boundary conditions, simulations were carried out and numerical results were compared with previous works of the other authors and the results of finite element method (FEM) analysis. The numerical results in terms of the nondimensional frequency parameters were presented over a range of relative thickness ratios.

2. Problem statement

Consider an isotropic, thick annular plate with outer radius r_o, inner radius r_i, minimum thickness $2h_i$, and maximum thickness $2h_o$ as depicted in Fig. 1. Clearly, $r_i = 0$ for a circular plate. The plate geometry and dimensions are defined in an orthogonal cylindrical coordinate system(r, θ, z). The Ritz method was used for derivation of the eigenvalue equations. Mathematical formulation is not detailed here due to space constraints. However, one may consult [8–10] for the underlying theory and step-by-step derivation of the formulation of the problem. Unless otherwise defined, nomenclature of Ref. [14] is adopted in this paper. The following parameters were also introduced:

$$R = \frac{r_i}{r_o}, \quad H = \frac{h_i}{h_o}, \quad \delta = \frac{h_o}{r_o}.$$

A code was developed for the solution of the associated eigenvalue problem. For all the calculations here, the Poisson's ratio has been taken as $v = 0.3$. The vibration frequency ω is expressed in terms of a nondimensional frequency parameter $\beta = \omega r_o \sqrt{\rho / G}$. The values in parentheses (n, s) in Tables 1–7 indicate that the vibrating mode has n nodal diameters and vibrates in the sth mode for the given n value. Different boundary conditions of the problem were defined by using basic radial functions in each of the displacement amplitude functions. For a detailed description of the basic radial functions for each of free edge, soft simple support, hard simple support, and clamped edge see [11]. By definition, only displacement along z-axis is fixed in soft simply supported plates. Displacements along both r and z axes are, instead, fixed in hard simply supported plates. In order to check the reliability and accuracy of the method presented in the preceding section, some convergence tests and comparison studies were performed.

Firstly, a convergence study is performed in Table 1 for the first nine frequency parameters of the annular plate with nonlinear thickness variation $p = 2$, different combinations of boundary conditions, i.e. free outer edge and clamped inner edge with inner–outer radius ratio $R = \frac{1}{5}$, length–radius ratio $\delta = \frac{1}{6}$ and inner–outer thickness ratio $H = \frac{1}{3}$.

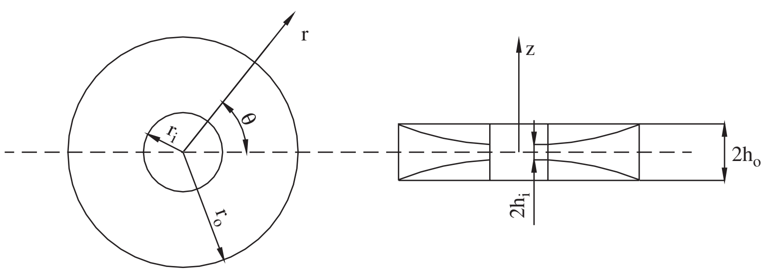

Fig. 1. Geometry and dimensions of a cross-section of a nonlinear thick annular plate in polar coordinates.

Table 1
Convergence of the first nine frequency parameters for annular plates with nonlinear thickness variation ($p = 2$) and different boundary conditions ($R = \frac{1}{5}$, $H = \frac{1}{3}$ and $\delta = \frac{1}{6}$)

Terms	Mode sequence number								
$N_1 \times N_2$	1	2	3	4	5	6	7	8	9
(A) *An annular plate with both outer and inner edges free* (*F–F*)									
5 × 4	0.4318 (2, 0)	0.6746 (0, 0)	1.109 (3, 0)	1.304 (1, 0)	1.557 (2, 1)	1.945 (4, 0)	2.312 (2, 2)	2.661 (0, 1)	2.698 (1, 1)
5 × 5	0.4318 (2, 0)	0.6746 (0, 0)	1.109 (3, 0)	1.304 (1, 0)	1.557 (2, 1)	1.945 (4, 0)	2.312 (2, 2)	2.661 (0, 1)	2.698 (1, 1)
6 × 4	0.4316 (2, 0)	0.6738 (0, 0)	1.109 (3, 0)	1.293 (1, 0)	1.554 (2, 1)	1.944 (4, 0)	2.299 (2, 2)	2.661 (0, 1)	2.698 (1, 1)
7 × 4	0.4315 (2, 0)	0.6734 (0, 0)	1.109 (3, 0)	1.289 (1, 0)	1.553 (2, 1)	1.944 (4, 0)	2.299 (2, 2)	2.661 (0, 1)	2.698 (1, 1)
8 × 4	0.4315 (2, 0)	0.6734 (0, 0)	1.109 (3, 0)	1.288 (1, 0)	1.553 (2, 1)	1.944 (4, 0)	2.299 (2, 2)	2.661 (0, 1)	2.698 (1, 1)
9 × 4	0.4315 (2, 0)	0.6734 (0, 0)	1.109 (3, 0)	1.287 (1, 0)	1.553 (2, 1)	1.944 (4, 0)	2.299 (2, 2)	2.661 (0, 1)	2.698 (1, 1)
(B) *An annular plate with free outer edge and hard simply supported inner edge* (F–S^h)									
5 × 4	0.1217 (1, 0)	0.2047 (0, 0)	0.4413 (2, 0)	1.076 (1, 1)	1.110 (3, 0)	1.487 (0, 1)	1.784 (1, 2)	1.945 (4, 0)	1.964 (2, 1)
6 × 4	0.1207 (1, 0)	0.2042 (0, 0)	0.4410 (2, 0)	1.073 (1, 1)	1.110 (3, 0)	1.483 (0, 1)	1.764 (1, 2)	1.944 (4, 0)	1.961 (2, 1)
7 × 4	0.1202 (1, 0)	0.2039 (0, 0)	0.4408 (2, 0)	1.072 (1, 1)	1.110 (3, 0)	1.480 (0, 1)	1.762 (1, 2)	1.944 (4, 0)	1.960 (2, 1)
8 × 4	0.1198 (1, 0)	0.2037 (0, 0)	0.4408 (2, 0)	1.071 (1, 1)	1.110 (3, 0)	1.479 (0, 1)	1.761 (1, 2)	1.944 (4, 0)	1.959 (2, 1)
9 × 4	0.1197 (1, 0)	0.2036 (0, 0)	0.4407 (2, 0)	1.071 (1, 1)	1.110 (3, 0)	1.479 (0, 1)	1.760 (1, 2)	1.944 (4, 0)	1.959 (2, 1)
(C) *An annular plate with free outer edge and clamped inner edge* (*F–C*)									
5 × 4	0.2025 (1, 0)	0.2830 (0, 0)	0.4558 (2, 0)	1.112 (3, 0)	1.233 (1, 1)	1.780 (0, 1)	1.945 (4, 0)	1.999 (1, 2)	2.055 (2, 1)
6 × 4	0.2018 (1, 0)	0.2826 (0, 0)	0.4550 (2, 0)	1.112 (3, 0)	1.232 (1, 1)	1.772 (0, 1)	1.945 (4, 0)	1.993 (1, 2)	2.054 (2, 1)
7 × 4	0.2014 (1, 0)	0.2823 (0, 0)	0.4544 (2, 0)	1.112 (3, 0)	1.231 (1, 1)	1.767 (0, 1)	1.945 (4, 0)	1.989 (1, 2)	2.053 (2, 1)
8 × 4	0.2012 (1, 0)	0.2821 (0, 0)	0.4541 (2, 0)	1.112 (3, 0)	1.231 (1, 1)	1.763 (0, 1)	1.945 (4, 0)	1.986 (1, 2)	2.053 (2, 1)
9 × 4	0.2011 (1, 0)	0.2820 (0, 0)	0.4540 (2, 0)	1.112 (3, 0)	1.231 (1, 1)	1.761 (0, 1)	1.945 (4, 0)	1.985 (1, 2)	2.053 (2, 1)

For the cases (a) and (b) in Table 1, frequency parameters have converged to four-digit accuracy using 9 terms in the radial direction and 4 terms in the thickness direction (i.e. 9×4 terms) of the Ritz polynomials. For the case (c), frequency parameters have converged to three-digit accuracy using the same terms (i.e. 9×4 terms) of the Ritz polynomials. Therefore, a rapid convergence has been obtained for an annular plate with parabolic thickness variation and free outer–inner edges (F–F) and free outer edge and hard simply supported inner edge (F–S), while for free outer edge and clamped inner edge (F–C), slightly higher terms are needed.

In Table 2, the accuracy of the first eight eigenfrequencies for circular and annular plates ($R = \frac{3}{10}$) with different boundary conditions are validated through comparison with the results presented by Zhou et al. [10] and Liew and yang [11]. For convenience in comparison, a new frequency parameter η is defined as $\eta = \omega r_o^2 \sqrt{\rho h_o / D}$ where $D(= E h_o^3/[12(1 - v^2)])$ is the flexural rigidity of the plate. In Table 3, the present solutions are compared with the published values of Kang [14] for completely free circular and annular plates ($R = \frac{1}{6}$) with linear ($p = 1$) and parabolic ($p = 2$) thickness variation for $(H, \delta) = (\frac{1}{4}, \frac{1}{6})$ and $(4, \frac{1}{24})$.

It is observed that the present solutions are in excellent agreement for all cases. The percent discrepancy is defined by

$$\text{Discrepancy } (\%) = \{[(\text{Kang [14] or FEM}) - \text{authors}]/\text{authors}\} \times 100.$$

Table 4 shows a comparison study of the present results with the converged finite element solutions obtained using the FEM. A well-known commercially available FEM package was used for extraction of the frequency parameters. Before proceeding to the cases for which frequency parameters are calculated for the first time, the package as well as the solution procedure were examined by solving some problems of the literature. It was determined that there is an excellent agreement between results of the FEM package and those of the literature. 3-D solid elements having all six degrees of freedom were adopted in all FEM calculations. A convergence study was conducted to ensure that the results of the calculations are independent

Table 2
Comparison of the first eight frequency parameters of circular and annular plates with different boundary conditions by the present solutions with those of Zhou et al. [10] and Liew and Yang [11,13]

Thickness δ	Source of reference	Mode sequence number							
		1	2	3	4	5	6	7	8
(A) *Circular plate with free boundary condition*									
0.05	Ref[a]	5.2791	8.8720	12.072	19.737	20.826	31.327	33.110	36.132
	Ref[b]	5.2795	8.8720	12.074	19.738	20.831	31.336	33.112	36.132
	Authors	5.2793	8.8720	12.073	19.737	20.828	31.330	33.111	36.132
0.15	Ref[a]	4.9005	8.0344	10.439	16.023	16.102	16.750	18.666	25.503
	Ref[b]	4.9005	8.0344	10.439	16.023	16.102	16.750	18.666	25.503
	Authors	4.9005	8.0344	10.439	16.023	16.102	16.750	18.666	25.503
(B) *Circular plate with hard simply supported boundary condition*									
0.05	Ref[b]	4.8975	13.580	23.725	24.555	28.310	37.472	44.900	49.626
	Authors	4.8975	13.580	23.722	24.555	28.310	37.472	44.900	49.624
0.15	Ref[b]	4.6234	7.9235	11.723	16.553	19.453	21.879	23.577	24.622
	Authors	4.6234	7.9234	11.721	16.553	19.452	21.879	23.577	24.622
(C) *Circular plate with clamped boundary condition*									
0.05	Ref[a]	9.9755	20.267	32.383	36.692	46.076	54.234	61.113	67.941
	Ref[b]	9.9909	20.297	32.430	36.744	46.140	54.308	61.186	67.969
	Authors	9.9908	20.297	32.430	36.743	46.139	54.310	61.185	67.970
0.15	Ref[a]	8.4606	15.442	22.654	22.715	25.134	26.176	30.078	34.222
	Ref[b]	8.4676	15.453	22.667	22.721	25.150	—	30.093	34.239
	Authors	8.4746	15.453	22.766	22.721	25.150	26.176^t	30.093	34.240
(D) *An annular plate with free outer edge and clamped inner edge* (*F–C*)									
0.1	Ref[a]	5.8498	6.1864	6.9366	9.7590	11.756	18.832	18.972	27.409
	Ref[c]	5.8662	6.2020	6.9482	—	11.769	18.833	18.973	—
	Authors	5.8656	6.2013	6.9478	9.7593^t	11.769	18.834	18.972	27.410
(E) *An annular plate with hard simply supported outer edge and free inner edge* (S^h–*F*)									
0.1	Ref[c]	4.5401	11.240	12.742	15.904	20.852	27.931	30.709	31.543
	Authors	4.5402	11.240	12.742	15.905	20.852	27.932	30.709	31.542
(F) *An annular plate with both outer and inner edges clamped* (*C–C*)									
0.1	Ref[a]	30.688	31.422	34.325	40.231	48.220	48.707	53.067	58.689
	Ref[c]	30.743	31.474	34.370	40.266	—	48.736	53.072	—
	Authors	30.743	31.474	34.369	40.257	48.220^t	48.736	53.071	58.695

0^t = torsional mode.
[a]Zhou et al. [10].
[b]Liew and Yang [13].
[c]Liew and Yang [11].

from the number of elements. The number of elements was about 14,000 in all of the solutions. Good agreements were found between the present results and the finite element solutions.

Having confirmed the convergence test as shown in Table 1 and having determined the high accuracy through the comparison study illustrated in Tables 2–4, on the basis of the present 3-D Ritz formulation, some numerical results are given for circular and annular plates of variable thickness with different combinations of boundary conditions in Tables 5–7.

In Table 5, the frequency parameters for the circular plates ($r_i = 0$) of variable thickness ($p = 0, 1, 2$) for $(H, \delta) = (1, \frac{1}{5})$, $(\frac{1}{3}, \frac{1}{6})$ and $(3, \frac{1}{18})$ with completely free, hard and soft simply supported and clamped edges are shown.

In Table 6, the results of frequency parameters for annular plates ($R = r_i/r_o = 1/5$) with both edges (outer and inner) free, soft simply supported and clamped are given with the same values of (p, H, δ) in Table 6.

Table 3
Comparison of the first five frequency parameters of completely free circular and annular plates with linear and parabolic thickness variation by the present solutions with Kang [14]

Mode number	$[p, H, \delta]$					
	$[1, \frac{1}{4}, \frac{1}{6}]$			$[2, \frac{1}{4}, \frac{1}{6}]$		
	Authors	Kang [14]	Discrepancy (%)	Authors	Kang [14]	Discrepancy (%)
($R = 0$) (A) *Circular plates with free boundary condition*						
1	0.5301	0.5301	0.0000	0.4410	0.4410	0.0000
2	0.8106	0.8106	0.0000	0.6678	0.6678	0.0000
3	1.301	1.301	0.000	1.129	1.129	0.000
4	1.732	1.732	0.000	1.358	1.358	0.000
5	1.936	1.936	0.000	1.794	1.794	0.000
($R = \frac{1}{6}$) (B) *Annular plates with free boundary condition*						
1	0.4932	0.4932	0.0000	0.4116	0.4116	0.0000
2	0.7613	0.7611	−0.0260	0.6343	0.6343	0.0000
3	1.242	1.242	0.000	1.071	1.071	0.000
4	1.490	1.489	−0.067	1.189	1.188	−0.084
5	1.621	1.621	0.000	1.542	1.542	0.000

Table 4
Comparison of the first nine frequency parameters of annular plates ($R = \frac{1}{5}$) with different boundary conditions between present solution and FEM

Mode number	$[p, H, \delta]$					
	$[1, 4, \frac{1}{18}]$			$[2, 4, \frac{1}{18}]$		
	Authors	FEM	Discrepancy (%)	Authors	FEM	Discrepancy (%)
(A) *Annular plates with hard simply supported outer edge and free inner edge* (S^h–F)						
1	0.5100	0.5109	−0.17	0.5924	0.5934	−0.17
2	1.210	1.220	−0.83	1.412	1.425	−0.92
3	2.051	2.055	−0.20	2.092	2.098	−0.29
4	2.138	2.143	−0.23	2.331	2.339	−0.34
5	2.615	2.623	−0.31	2.933	2.941	−0.27
6	2.913	2.922	−0.30	2.951	2.960	−0.31
7	3.036	3.050	−0.46	3.346	3.352	−0.18
8	3.406	3.417	−0.32	3.381	3.393	−0.36
9	3.467	3.476	−0.26	3.751	3.766	−0.40
(B) *Annular plates with both outer and inner edges hard simply supported* (S^h–S^h)						
1	1.575	1.579	−0.25	1.768	1.770	−0.11
2	1.586	1.591	−0.31	1.807	1.811	−0.22
3	2.110	2.122	−0.57	2.104	2.111	−0.33
4	2.153	2.161	−0.31	2.396	2.401	−0.13
5	2.914	2.920	−0.21	2.952	2.963	−0.27
6	3.043	3.055	−0.40	3.367	3.375	−0.24
7	3.996	4.001	−0.13	3.986	3.998	−0.30
8	4.179	4.189	−0.24	4.233	4.245	−0.28

In Table 7, the frequency parameters for annular plates ($R = r_i/r_o = 1/5$) with hard and soft simply supported outer edge and free inner edge (S–F), clamped outer edge and free inner edge (C–F), clamped outer edge and soft simply supported inner edge (C–S) and soft simply supported outer edge and clamped inner edge (S–C) are given with the same values of (p, H, δ) in Tables 5 and 6.

Table 5
Results for the first nine frequency parameters β for non-uniform circular plates ($r_i = 0$) with different boundary conditions

Mode number	$[p, H, \delta]$				
	$[0, 1, \frac{1}{5}]$	$[1, \frac{1}{3}, \frac{1}{6}]$	$[1, 3, \frac{1}{18}]$	$[2, \frac{1}{3}, \frac{1}{6}]$	$[2, 3, \frac{1}{18}]$
(A) *Circular plates with free boundary condition*					
1	0.9078 (2, 0)	0.5528 (2, 0)	0.6377 (2, 0)	0.4685 (2, 0)	0.7646 (2, 0)
2	1.462 (0, 0)	0.8553 (0, 0)	1.058 (0, 0)	0.7184 (0, 0)	1.271 (0, 0)
3	1.860 (3, 0)	1.336 (3, 0)	1.142 (3, 0)	1.177 (3, 0)	1.404 (3, 0)
4	2.346 (2, 1)	1.834 (1, 0)	1.722 (4, 0)	1.503 (1, 0)	2.096 (4, 0)
5	2.731 (1, 0)	2.001 (2, 1)	1.950 (1, 0)	1.895 (2, 1)	2.361 (1, 0)
6	2.780 (1, 1)	2.256 (4, 0)	2.774 (2, 1)	2.040 (4, 0)	2.755 (2, 1)
7	2.889 (4, 0)	2.719 (1, 1)	2.810 (1, 1)	2.565 (2, 2)	2.860 (1, 1)
8	3.436 (0, 1)	3.018 (2, 2)	2.952 (2, 2)	2.600 (0, 1)	3.457 (2, 2)
9	3.600 (3, 1)	3.097 (0, 1)	3.319 (0, 1)	2.980 (0, 2)	3.821 (0, 1)
(B) *Circular plates with soft simply supported boundary condition*					
1	0.8638 (0, 0)	0.5414 (0, 0)	0.4797 (0, 0)	0.4704 (0, 0)	0.5768 (0, 0)
2	2.059 (1, 0)	1.416 (1, 0)	1.162 (1, 0)	1.185 (1, 0)	1.380 (1, 0)
3	2.346 (2, 0)	2.002 (2, 0)	2.030 (2, 0)	1.896 (2, 0)	2.305 (2, 0)
4	2.743 (1, 1)	2.457 (2, 1)	2.491 (0, 1)	2.134 (2, 1)	2.756 (2, 1)
5	3.268 (2, 1)	2.723 (0, 1)	2.774 (2, 1)	2.261 (0, 1)	2.792 (0, 1)
6	3.440 (0, 1)	2.729 (1, 1)	2.813 (1, 1)	2.694 (1, 1)	2.863 (1, 1)
7	3.605 (3, 0)	3.102 (0, 2)	2.897 (3, 0)	2.986 (0, 2)	3.269 (3, 0)
8	3.707 (0, 2)	3.255 (3, 0)	3.698 (1, 2)	3.083 (3, 0)	3.916 (0, 2)
9	4.261 (2, 2)	3.596 (3, 1)	3.799 (4, 0)	3.203 (3, 1)	4.118 (1, 2)
(C) *Circular plates with clamped boundary condition*					
1	1.486 (0, 0)	1.278 (0, 0)	0.7217 (0, 0)	1.189 (0, 0)	0.8008 (0, 0)
2	2.595 (1, 0)	2.143 (1, 0)	1.507 (1, 0)	1.977 (1, 0)	1.688 (1, 0)
3	3.332 (1, 1)	3.118 (2, 0)	2.436 (2, 0)	2.892 (2, 0)	2.658 (2, 0)
4	3.715 (2, 0)	3.325 (0, 1)	2.751 (1, 1)	3.008 (0, 1)	2.732 (1, 1)
5	3.832 (0^t, 0)	3.832 (1, 1)	2.751 (1, 1)	3.140 (0, 1)	2.732 (1, 1)
6	4.096 (0, 1)	4.182 (3, 0)	3.343 (0^t, 0)	3.898 (3, 0)	3.140 (0, 1)
7	4.855 (3, 0)	4.183 (0^t, 0)	3.347 (3, 0)	3.995 (1, 1)	3.271 (0^t, 0)
8	5.177 (2, 1)	4.500 (1, 2)	4.140 (1, 2)	4.108 (1, 2)	3.272 (3, 0)
9	5.370 (1, 2)	5.292 (4, 0)	4.279 (4, 0)	4.370 (0^t, 0)	4.477 (1, 2)
10	5.471 (1, 3)	5.294 (2, 1)	4.922 (2, 1)	5.267 (2, 1)	4.748 (2, 1)
(D) *Circular plates with hard simply boundary condition*					
1	0.8638 (0, 0)	0.5414 (0, 0)	0.4797 (0, 0)	0.4704 (0, 0)	0.5768 (0, 0)
2	1.161 (1, 0)	1.257 (1, 0)	1.017 (1, 0)	1.210 (1, 0)	1.008 (1, 0)
3	2.081 (1, 1)	1.437 (1, 1)	1.164 (1, 1)	1.294 (1, 1)	1.382 (1, 1)
4	2.424 (2, 0)	2.084 (2, 0)	2.038 (2, 0)	2.001 (2, 0)	2.312 (2, 0)
5	3.327 (2, 1)	2.515 (2, 1)	2.491 (0, 1)	2.199 (2, 1)	2.312 (2, 0)
6	3.440 (0, 1)	2.723 (0, 1)	2.835 (2, 1)	2.261 (0, 1)	2.801 (2, 1)
7	3.605 (3, 0)	3.102 (0, 2)	2.915 (3, 0)	2.986 (0, 2)	3.271 (0^t, 0)
8	3.707 (0, 2)	3.256 (3, 0)	3.700 (1, 2)	3.083 (3, 0)	3.284 (3, 0)
9	3.832 (0^t, 0)	3.690 (3, 1)	3.829 (4, 0)	3.306 (3, 1)	4.119 (1, 2)

0^t = torsional mode.

3. Conclusions

Based on the small strain and linear elasticity theory, an exhaustive study of 3-D free vibration of thick, circular and annular plates with linear and parabolic thickness variation and different combinations inner and outer boundary conditions has been performed. The influences of the plate thickness ratios, linear and parabolic thickness variation and different boundary conditions on the frequency parameters are examined. The Ritz method is applied to derive the eigenvalue equation. The excellent accuracy of the method has been

Table 6
Results for the first nine frequency parameters β for non-uniform circular plates ($R = \frac{1}{5}$) with different boundary conditions

Mode number	$[p, H, \delta]$				
	$[0, 1, \frac{1}{5}]$	$[1, \frac{1}{3}, \frac{1}{6}]$	$[1, 3, \frac{1}{18}]$	$[2, \frac{1}{3}, \frac{1}{6}]$	$[2, 3, \frac{1}{18}]$
(A) *Annular plates with both outer and inner edges free (F–F)*					
1	0.8546 (2, 0)	0.5071 (2, 0)	0.6399 (2, 0)	0.4315 (2, 0)	0.7367 (2, 0)
2	1.381 (0, 0)	0.7927 (0, 0)	1.094 (0, 0)	0.6734 (0, 0)	1.239 (0, 0)
3	1.851 (3, 0)	1.267 (3, 0)	1.255 (3, 0)	1.109 (3, 0)	1.500 (3, 0)
4	1.880 (2, 1)	1.544 (1, 0)	1.875 (4, 0)	1.288 (1, 0)	2.091 (2, 1)
5	2.498 (1, 0)	1.596 (2, 1)	2.000 (1, 0)	1.553 (2, 1)	2.257 (4, 0)
6	2.787 (1, 1)	2.171 (4, 0)	2.137 (2, 1)	1.944 (4, 0)	2.282 (1, 0)
7	2.889 (4, 0)	2.713 (0, 1)	2.906 (1, 1)	2.299 (2, 2)	2.943 (1, 1)
8	3.031 (0, 1)	2.743 (1, 1)	3.065 (2, 2)	2.661 (0, 1)	3.346 (0, 1)
9	3.502 (3, 1)	3.064 (3, 1)	3.405 (0, 1)	2.698 (1, 1)	3.558 (2, 2)
(B) *Annular plates with both outer and inner edges soft simply supported (S–S)*					
1	2.253 (0, 0)	1.358 (0, 0)	1.575 (1, 0)	1.124 (0, 0)	1.768 (0, 0)
2	2.468 (1, 0)	1.602 (2, 0)	1.586 (0, 0)	1.370 (1, 0)	1.807 (1, 0)
3	2.800 (1, 1)	1.633 (1, 0)	2.110 (2, 0)	1.558 (2, 0)	2.104 (2, 0)
4	3.054 (0, 1)	2.383 (2, 1)	2.153 (2, 1)	2.049 (2, 1)	2.396 (2, 1)
5	3.294 (2, 0)	2.722 (0, 1)	2.914 (1, 1)	2.670 (0, 1)	2.952 (1, 1)
6	3.510 (3, 0)	2.755 (1, 1)	3.043 (3, 0)	2.709 (1, 1)	3.367 (0, 1)
7	4.081 (2, 1)	3.072 (3, 0)	3.431 (0, 1)	2.939 (3, 0)	3.390 (3, 0)
8	4.466 (3, 1)	3.429 (3, 1)	3.996 (4, 0)	3.010 (3, 1)	3.986 (3, 1)
9	4.686 (4, 0)	4.058 (2, 2)	4.179 (2, 2)	4.017 (2, 2)	4.233 (2, 2)
(C) *Annular plates with both outer and inner edges clamped (C–C)*					
1	3.138 (0, 0)	2.640 (0, 0)	2.380 (0, 0)	2.602 (0, 0)	2.404 (0, 0)
2	3.306 (1, 0)	2.779 (1, 0)	2.427 (1, 0)	2.715 (1, 0)	2.484 (1, 0)
3	3.917 (2, 0)	3.273 (2, 0)	2.789 (2, 0)	3.114 (2, 0)	2.937 (2, 0)
4	4.236 (0^t, 0)	4.106 (3, 0)	3.533 (3, 0)	3.842 (3, 0)	3.649 (0^t, 0)
5	4.722 (1, 1)	4.571 (0^t, 0)	3.760 (0^t, 0)	4.798 (0^t, 0)	3.779 (3, 0)
6	4.893 (3, 0)	4.978 (1, 1)	4.317 (1, 1)	4.799 (4, 0)	4.175 (1, 1)
7	5.740 (2, 1)	5.155 (4, 0)	4.445 (4, 0)	5.222 (1, 1)	4.776 (4, 0)
8	6.009 (4, 0)	5.588 (0, 1)	5.280 (0, 1)	5.349 (0, 1)	5.245 (2, 1)
9	6.328 (0, 1)	5.727 (1, 2)	5.452 (2, 1)	5.470 (1, 2)	5.439 (0, 1)

0^t = torsional mode.

demonstrated by the convergence and comparison studies. In the present analysis, a set of orthogonal polynomial series multiplying by a boundary function was adopted as the admissible functions of each displacement component. Thorough convergence studies shown in Table 1 have been made, which indicate that the benchmark frequency values given in Tables 5–7 have converged to at least three significant figures. The first nine frequency parameters of thick circular and annular plates with different boundary conditions were presented for the first time. The plates are considered to have either a constant thickness or variable thickness. For the case of variable thickness, linear and parabolic thickness variations were considered. The frequency parameters were given for inner–outer radius ratio $R = \frac{1}{5}$ and different combinations of length to radius and inner to outer thickness ratios. For circular plates, frequency parameters were given for four different types of boundary conditions, i.e. free, clamped, hard simply supported, and soft simply supported. For annular plates, frequency parameters were given for seven combinations of boundary conditions, i.e. both outer and inner edges free, both outer and inner edges soft simply supported, both outer and inner edges clamped, softly simply supported outer edge and free inner edge, clamped outer edge and free inner edge, clamped outer edge and soft simply supported inner edge, and hard simply supported outer edge and free inner edge.

After the validation of the present results (convergence and comparison) with the available analytical solutions, and also the finite element solutions for some problems in the literature, the results in Tables 1 and

Table 7
Results for the first nine frequency parameters β for non-uniform circular plates ($R = \frac{1}{5}$) with different boundary conditions

Mode number	$[p, H, \delta]$				
	$[0, 1, \frac{1}{5}]$	$[1, \frac{1}{3}, \frac{1}{6}]$	$[1, 3, \frac{1}{18}]$	$[2, \frac{1}{3}, \frac{1}{6}]$	$[2, 3, \frac{1}{18}]$
(A) *Annular plates with soft simply supported outer edge and free inner edge (S–F)*					
1	0.8335 (0, 0)	0.5095 (0, 0)	0.5100 (0, 0)	0.4475 (0, 0)	0.5924 (0, 0)
2	1.921 (1, 0)	1.227 (1, 0)	1.210 (1, 0)	1.039 (1, 0)	1.412 (1, 0)
3	2.799 (1, 1)	1.598 (2, 0)	2.051 (2, 0)	1.555 (2, 0)	2.092 (2, 0)
4	3.035 (0, 1)	2.267 (2, 1)	2.138 (2, 1)	1.943 (2, 1)	2.331 (2, 1)
5	3.192 (2, 0)	2.718 (0, 1)	2.615 (0, 1)	2.342 (0, 1)	2.933 (0, 1)
6	3.507 (3, 0)	2.755 (1, 1)	2.913 (1, 1)	2.667 (0, 2)	2.951 (1, 1)
7	3.951 (0, 2)	2.953 (0, 2)	3.036 (3, 0)	2.709 (1, 1)	3.346 (0, 2)
8	4.078 (2, 1)	3.070 (3, 0)	3.406 (0, 2)	2.937 (3, 0)	3.381 (3, 0)
9	4.450 (3, 1)	3.404 (3, 1)	3.467 (1, 2)	2.938 (1, 2)	3.751 (1, 2)
(B) *Annular plates with clamped outer edge and free inner edge (C–F)*					
1	1.541 (0, 0)	1.394 (0, 0)	0.7499 (0, 0)	1.240 (0, 0)	0.8150 (0, 0)
2	2.444 (1, 0)	1.958 (1, 0)	1.518 (1, 0)	1.781 (1, 0)	1.672 (1, 0)
3	3.582 (1, 1)	2.943 (2, 0)	2.423 (2, 0)	2.692 (2, 0)	2.643 (2, 0)
4	3.621 (2, 0)	3.681 (0, 1)	2.908 (1, 1)	3.258 (0, 1)	2.917 (1, 1)
5	3.861 (0^t, 0)	4.016 (3, 0)	3.024 (0, 1)	3.702 (1, 1)	3.267 (0^t, 1)
6	4.302 (0, 1)	4.104 (1, 1)	3.298 (0^t, 0)	3.703 (3, 0)	3.272 (0, 1)
7	4.337 (2, 1)	4.212 (1, 2)	3.451 (3, 0)	4.336 (1, 2)	3.718 (3, 0)
8	4.820 (3, 0)	4.332 (0^t, 1)	3.794 (2, 1)	4.492 (0^t, 0)	3.767 (2, 1)
9	4.910 (1, 2)	4.856 (2, 1)	3.803 (1, 2)	4.767 (4, 0)	4.020 (1, 2)
(C) *Annular plates with clamped outer edge and soft simply supported inner edge (C–S)*					
1	2.932 (0, 0)	2.383 (0, 0)	1.966 (1, 0)	2.276 (0, 0)	2.093 (0, 0)
2	3.056 (1, 0)	2.560 (1, 0)	1.968 (0, 0)	2.419 (1, 0)	2.138 (1, 0)
3	3.581 (1, 1)	3.139 (2, 0)	2.500 (2, 0)	2.918 (2, 0)	2.722 (2, 0)
4	3.740 (2, 0)	4.060 (3, 0)	2.907 (1, 1)	3.757 (3, 0)	2.916 (1, 1)
5	3.861 (0^t, 0)	4.213 (1, 1)	3.298 (0^t, 1)	4.336 (1, 1)	3.267 (0^t, 0)
6	4.354 (2, 1)	4.332 (0^t, 0)	3.458 (3, 0)	4.492 (0^t, 0)	3.740 (3, 0)
7	4.839 (3, 0)	4.869 (2, 1)	3.816 (2, 1)	4.776 (4, 0)	3.783 (2, 1)
8	5.580 (1, 2)	5.142 (4, 0)	4.439 (4, 0)	4.970 (0, 1)	4.776 (4, 0)
9	5.634 (0, 1)	5.315 (0, 1)	5.084 (0, 1)	5.014 (0, 1)	5.094 (1, 2)
(D) *Annular plates with hard simply supported outer edge and free inner edge (S^h–F)*					
1	0.8335 (0, 0)	0.5095 (0, 0)	0.5100 (0, 0)	0.4475 (0, 1)	0.5924 (0, 0)
2	1.200 (1, 0)	1.250 (1, 0)	1.048 (1, 0)	1.064 (1, 0)	1.049 (1, 0)
3	1.914 (2, 0)	1.307 (1, 1)	1.212 (1, 1)	1.336 (1, 1)	1.414 (1, 1)
4	1.942 (1, 1)	1.658 (2, 0)	2.059 (2, 0)	1.641 (2, 0)	2.097 (2, 0)
5	3.035 (0, 1)	2.327 (2, 1)	2.144 (2, 1)	2.009 (2, 1)	2.338 (2, 1)
6	3.249 (2, 1)	2.718 (0, 1)	2.615 (0, 1)	2.342 (0, 1)	2.933 (0, 1)
7	3.509 (3, 0)	2.953 (0, 2)	3.053 (3, 0)	2.667 (0, 2)	3.267 (0^t, 0)
8	3.861 (0^t, 0)	3.072 (3, 0)	3.298 (0^t, 0)	2.937 (3, 0)	3.346 (0, 2)
9	3.951 (0, 3)	3.500 (3, 1)	3.406 (0, 2)	2.946 (1, 2)	3.396 (3, 0)

0^t = torsional mode.

5–7 can serve as benchmark solutions for researchers to validate their numerical methods (i.e. classical, 3-D and 3-D theories) and for designers to use such plates in their structures in the future.

References

[1] K.M. Liew, J.B. Han, Z.M. Xiao, Vibration analysis of circular Mindlin plates using the differential quadrature method, *Journal of Sound Vibration* 205 (1997) 617–630.

[2] K.M. Liew, S. Kitipornchai, C.W. Lim, Free vibration analysis of thick superelliptical plates, *Journal of Engineering Mechanics* 124 (1998) 137–145.
[3] K.M. Liew, Y. Xiang, S. Kitipornchai, Research on thick plate vibration: a literature survey, *Journal of Sound and Vibration* 180 (1995) 163–176.
[4] O.K. Aksentian, T.N. Selezneva, Determination of frequencies of natural vibrations of circular plates, *Journal of Applied Mathematics and Mechanics* 40 (1976) 96–103.
[5] T. Irie, G.Y. Amada, S. Aomura, Free vibrations of a Mindlin annular plate of varying thickness, *Journal of Sound and Vibration* 66 (1979) 187–197.
[6] J.R. Hutchinson, Vibrations of thick free circular plates, exact versus approximate solutions, *Journal of Applied Mechanics* 51 (1984) 581–585.
[7] J.R. Hutchinson, S.A. EI-Azhari, On the vibration of thick annular plates. Proceedings of Euromechanical Colloquium 219, 1986, Refined Dynamical Theories of Beams, Plates and Shells and Their Applications, Eds. H. Irretier and I. Elishakoff, Springer Verlag, Berlin, pp. 102–111, 1987.
[8] J. So, A.W. Leissa, Three-dimensional vibrations of thick circular and annular plates, *Journal of Sound and Vibration* 209 (1998) 15–41.
[9] J.H. Kang, A.W. Leissa, Three-dimensional vibrations of thick, linearly tapered, annular plates, *Journal of Sound and Vibration* 217 (1998) 927–944.
[10] D. Zhou, F.T.K. Au, Y.K. Cheung, S.H. Lo, Three-dimensional vibration analysis of circular and annular plates via the Chebyshev–Ritz method, *International Journal of Solids and Structures* 40 (2003) 3089–3105.
[11] K.M. Liew, B. Yang, Elasticity solutions for free vibrations of annular plates from three-dimensional analysis, *International Journal of Solids and Structures* 37 (2000) 7689–7702.
[12] C.F. Liu, Y.T. Lee, Finite element analysis of three-dimensional vibrations of thick circular and annular plates, *Journal of Sound and Vibration* 233 (2000) 63–80.
[13] K.M. Liew, B. Yang, Three-dimensional elasticity solutions for free vibrations of circular plates: a polynomials-Ritz analysis, *Computer Methods in Applied Mechanics and Engineering* 175 (1999) 189–201.
[14] J.H. Kang, Three-dimensional vibration analysis of thick, circular and annular plates with nonlinear thickness variation, *Computers & Structures* 81 (2003) 1663–1675.